On semiclassical calculation of three-point functions in $AdS_5 \times T^{1,1}$

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Abstract

Recently there has been progress on the computation of two- and three-point correlation functions with two "heavy" states via semiclassical methods. We extend this analysis to the case of $AdS_5 \times T^{1,1}$, and examine the suggested procedure for the case of several simple string solutions. By making use of AdS/CFT duality, we derive the relevant correlation functions of operators belonging to the dual gauge theory.

1 Introduction

One of the most interesting outputs of theoretical physics of last years was the conjectured duality between the string theory on $AdS_5 \times S^5$ and $N = 4$ Super Yang-Mills field theory [1]. Shortly after Maldacena's paper there were many checks of this conjecture. Due to the complicated Green-Schwartz superstring action on curved spacetime, most of them have been worked out in the supergravity (SUGRA) approximation. In order to check the duality, one usually compares various observables e.g. scaling dimensions, correlation functions. These quantities depend in general non-trivially on the t'Hooft coupling $\lambda = g_{YM}^2 N$. A perturbative description of these observables covers however opposite regions of the coupling λ, which makes it difficult to check the duality. Fortunately there is a small class of operators, the BPS operators, which have due to the supersymmetry trivial dependence on λ. They allow to compare both sides of the duality. The challenge is however, to go beyond the BPS case. One way is to consider special limits of parameters other than λ as in the BMN case [2]. A significant simplification comes when one focuses on a particular class of observables - the spectrum of energies of single string states. They correspond to the scaling dimensions of the single trace operators on the field theory side. The idea of BMN relies on the fact that for particular string states with large quantum numbers there are new limits where certain quantum corrections are suppressed. In the BMN case (for reviews see [3, 4, 5]) one considers a small closed string moving with large angular momentum $J \gg 1$ around a great circle of S^5. It represents a particular semi-classical sector of near BPS states, which allows in the limit $J \to \infty$, $\frac{\lambda}{J^2}$ − fixed for an exact correspondence between energies of the string states and scaling dimensions of the SYM operators [2]. This is possible since one can interpret the states semiclassically as quadratic corrections to a point-like string running along the great circle (the geodesic) of the S^5. In this case the corrections higher than one-loop are suppressed in the limit $J \to \infty$, $\frac{\lambda}{J^2}$ − fixed [6, 7] . For other, far from BPS, semiclassical sectors of string states

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we can still hope for a similar behavior. In [8] it was noticed that for string states with at least one large S^5 spin component J the classical energy has expansion in powers of $\frac{\lambda}{J^2}$ and that the quantum superstring corrections are suppressed in the limit $J \to \infty$, $\frac{\lambda}{J^2}$ − fixed. In this limit we can thus use (semi)classical approach knowing that the energy goes as $E = J + \ldots$. Precise tests of the AdS/CFT correspondence in a non-BPS sector were achieved [9, 10, 11, 12, 13] comparing the expansion of the classical string energy with the perturbative SYM anomalous dimension. This is however only half of the story, since in order to describe a CFT one needs to know the spectrum and the three point correlation functions of primary operators. As we note above, much is known about the spectrum of conformal weights. The situation with the three-point functions is however different and there is still a lot to reveal. Since the conformal invariance determines the spacetime dependence of the three-point correlation functions the essential information is stored in their couplings. Even in the well studied $N = 4$ SYM case we still do not know much about the *generic* three-point couplings. It is due to the lack of a tool for calculation of the three-point correlation functions with massive string states. The path integral approach to compute the string theory partition function for heavy string states presented in [14] opened a new way to compute the two-point function at strong coupling for the cases presented in [15, 8, 16]. Recently a method to calculate the three-point function at strong coupling corresponding to two heavy string states and a (light) supergravity field was presented [18, 17]. The method was applied to various cases of known string solutions in $AdS_5 \times S^5$ [17]-[27], as well as to $AdS_4 \times CP^3$ [28] and less supersymmetric cases [29].

In this short note we extend this successful in $AdS_5 \times S^5$ background idea to a less supersymmetric case. Our background is the AdS_5 times the Sasaki-Einstein space $T^{1,1}$. We calculate the three-point functions of certain gauge theory operators at strong coupling dual to various simple string solutions.

2 The two-point functions and the method

The study of the holography for less supersymmetric theories was initiated in [30] where a stack of D3 branes was putted on the tip of conifold. The geometry of such backgrounds was described for instance in [31, 32]. In this section we briefly describe the method and setup our notations for strings in $AdS_5 \times T^{1,1}$ background. In order to keep track with the already know results we use a more general metric of the squashed sphere X_5 rather than the $T^{1,1}$. This setup allows us to turn to the case of the sphere S^5 or the Sasaki-Einstein space $T^{1,1}$ simply by a particular choice of parameters. For convenience we use Poincaré coordinates to parametrize the AdS_5 space, since the boundary is given by $z = 0$. The metric ds^2 for the target space $AdS_5 \times X_5$ is thus of the form $ds^2 = ds_X^2 + ds_A^2$ with the X_5 and the AdS_5 part defined as follows :

$$ds_X^2 = a\left(d\theta_1^2 + \sin^2(\theta_1)\, d\phi_1^2 + d\theta_2^2 + \sin^2(\theta_2)\, d\phi_2^2\right) + b\left(d\psi + \cos(\theta_1)\, d\phi_1 + \cos(\theta_2)\, d\phi_2\right)^2 \tag{1}$$

$$ds_A^2 = \frac{dx^\mu dx_\mu + dz^2}{z^2} \tag{2}$$

For two special values of the parameters a, b , namely $a = 1/4$, $b = 1/4$ and $a = 1/6$, $b = 1/9$ the metric (1) corresponds to S^5 and $T^{1,1}$ respectively. For both, the target space and the worldsheet we use Minkowski signature. The action we work with is the standard Polyakov action

$$S[X,\Phi] = -\frac{\sqrt{\lambda}}{4\pi}\int \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} h^{ab} d\sigma d\tau e^{\frac{\phi}{2}} + \ldots = -\frac{\sqrt{\lambda}}{4\pi}\int \left(\mathcal{L}_X + \mathcal{L}_A\right) d\sigma d\tau \tag{3}$$

where $G_{\mu\nu}$ is the target space metric, ϕ represents fluctuations of the dilaton field and dots contain other fields such as the B-field and fermionic terms. We consider trivial topology of the worldsheet and use the conformal gauge i.e. the 2-dimensional metric is just $h^{ab} = diag(-1, 1)$.

Our approach is based on a recent proposal by Costa et al. [18] for the calculation of the three-point function at strong coupling. We will shortly review their method to set up the notation.

Let us first recall the two-point correlation function of operators dual to classical string states at strong coupling as described in [14] .

As a warm-up example, let us start with a point-like string, which we distinguish with the (sub)subscript P when necessary. The action (3) for the point-particle case has the form

$$S_P[X,\mathbf{e},\Phi] = \frac{1}{2}\int_0^1 d\tau e^{\frac{\phi}{4}}\left(\mathbf{e}^{-1}\partial_\tau X^\mu \partial_\tau X^\nu G_{\mu\nu} - \mathbf{e}m^2\right) \tag{4}$$

where ϕ represents fluctuations of the dilaton field and $\mathbf{e} = \mathbf{e}(\tau)$ is the einbein. For convenience one passes from the einbein formulation to the modular parameter s (see [14] for details). The action (4) on the AdS vacuum takes the form

$$S_P[X,s] \equiv S_P[X,s,\Phi=0] = \frac{1}{2}\int_{-\frac{s}{2}}^{\frac{s}{2}} d\tau \mathcal{L}_P = -\frac{1}{2}\int_{-\frac{s}{2}}^{\frac{s}{2}}\left(m^2 - \frac{x'^2(\tau)+z'^2(\tau)}{z^2(\tau)}\right)d\tau$$

where we assume that the particle is moving along a direction x and that all distances on the boundary are space-like. A generic solution to the equations of motion

$$\begin{aligned} x(\tau) =& R\tanh(\kappa\tau) + x_0 \\ z(\tau) =& \frac{R}{\cosh(\kappa\tau)} \end{aligned} \tag{5}$$

includes three free parameters R, κ, x_0 , which are further fixed by the following boundary conditions

$$x\left(-\frac{s}{2}\right) = 0\,,\ z\left(-\frac{s}{2}\right) = \epsilon\,,\ x\left(\frac{s}{2}\right) = x_f\,,\ z\left(\frac{s}{2}\right) = \epsilon. \tag{6}$$

Imposing conditions (6) one finds

$$z\left(\pm\frac{s}{2}\right) = \frac{R}{\cosh\left(\kappa\frac{s}{2}\right)} = \epsilon.$$

In the limit $\epsilon \ll 1$, we approximately get $x_0 \sim R$ and $R \sim \frac{x_f}{2}$ which results in the following relation for the parameter κ

$$\kappa \approx \frac{2}{s}\log\left(\frac{x_f}{\epsilon}\right). \tag{7}$$

Putting all these relations to the action we obtain

$$S_P\left[\bar{X},s\right] = \frac{1}{2}s\left(\kappa^2 - m^2\right) = \frac{1}{2}s\left(\frac{4\log^2\left(\frac{x_f}{\epsilon}\right)}{s^2} - m^2\right)$$

where the second equality comes after the substitution of (7). With the value of the saddle point[1]

$$\bar{s} = -\imath\frac{2\log\left(\frac{x_f}{\epsilon}\right)}{m} \tag{8}$$

we compute the 2-point function of the following form

$$\langle\mathcal{O}(0),\mathcal{O}(x_f)\rangle \approx e^{iS_P\left[\bar{X},\bar{s}\right]} = \left(\frac{x_f}{\epsilon}\right)^{-2m}. \tag{9}$$

Before we go on with the 3-point function let us recall that in order to obtain the correct scaling of the two-point correlation function for the operator $\mathcal{O}_A$ dual to a heavy string field one has to convolute the generating functional with the wave function as described in [14] . Practically

[1] Since we are working in Minkowski signature we get an imaginary saddle point.

it boils down to a change of the measure in the string path integral, such that we work with the effective action

$$\tilde{S}[X,s] = S[X,s] - \bar{S}[X,s] = S[X,s] - \int_{-\frac{s}{2}}^{\frac{s}{2}} d\tau \int_0^{2\pi} d\sigma \left((\Pi - \Pi_0)^a \left(\dot{X} - \dot{X}_0 \right)_a + \Pi^i \dot{X}_i \right) \tag{10}$$

$$\Pi_0^a = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \Pi^a(\tau,\sigma) \ , \ \dot{X}_0^a = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \dot{X}^a(\tau,\sigma)$$

where Π are the worldsheet canonical momenta. The sub/super scripts $\{a,i\}$ are indices for the AdS and X_5 parts respectively.

The 3-point correlation function $\langle \mathcal{O}_A(x_i)\, \mathcal{O}_A(x_f)\, \mathcal{D}_\chi(y) \rangle$ with $\mathcal{O}_A$ being the operator corresponding to a heavy string field and $\mathcal{D}_\chi$ a chiral operator corresponding to a light supergravity field can be approximated [18] as follows

$$\langle \mathcal{O}_A(x_i)\, \mathcal{O}_A(x_f)\, \mathcal{D}_\chi(y) \rangle \approx \frac{I_\chi\left[\bar{X}, \bar{s}, y\right]}{|x_i - x_f|^{2\Delta_A}}. \tag{11}$$

Here Δ_A is the dimension of the operator $\mathcal{O}_A$; x_i , x_f are the insertion points of the operators $\mathcal{O}_A$, y is the insertion point of the operator $\mathcal{D}_\chi$ and I_χ is the interaction term between heavy string fields and the supergravity field χ . This interaction term is of the form

$$I_\chi[X,s,y] = \imath \int_{-\frac{s}{2}}^{\frac{s}{2}} d\tau \int_0^{2\pi} d\sigma \left. \frac{\delta S[X,s,\Phi]}{\delta\chi} \right|_{\Phi=0} K_\chi(X(\tau,\sigma), y)$$

with $K_\chi(X(\tau,\sigma),y)$ being the bulk-to-boundary propagator for the field χ. We introduce for convenience the integrand i_χ such that $I_\chi = \int_{\frac{s}{2}}^{\frac{s}{2}} \int_0^{2\pi} i_\chi d\tau d\sigma$. In the case of the dilaton field $(\chi = \phi)$ and the action (3) we thus have

$$i_\phi = \frac{\imath}{2} \mathcal{L} K_\phi = \frac{3}{2\pi^2} \mathcal{L} \left(\frac{z}{z^2 + (x-y)^2} \right)^4, \tag{12}$$

where $\mathcal{L} = \mathcal{L}_A + \mathcal{L}_X$ is the Lagrangian. Calculating the interaction term for the point particle and taking the leading order approximation in the parameter ϵ we get

$$I_{\phi_P} \approx -\frac{\imath}{32\pi^2 \log\left(\frac{x_f}{\epsilon}\right) s} \left(s^2 m^2 - 4\log^2\left(\frac{x_f}{\epsilon}\right) \right) \frac{x_f^4}{(x_f - y)^4\, y^4}.$$

Evaluating the above interaction term at the saddle point (8) we end up with

$$I_{\phi_P} \approx -\frac{m}{8\pi^2} \frac{x_f^4}{(x_f - y)^4\, y^4},$$

which gives a 3-point function of the form

$$\langle \mathcal{O}_A(0)\, \mathcal{O}_A(x_f)\, \mathcal{L}(y) \rangle \approx -\frac{m}{8\pi^2} \frac{1}{x_f^{2\Delta_A - 4} y^4 (x_f - y)^4}. \tag{13}$$

We can check the consistency of our result with the expectation from the renormalization group as it was proposed in [18]. For the coupling $a_{\mathcal{L}AA}$ we obtain

$$2\pi^2 a_{\mathcal{L}AA} = -\lambda \frac{\partial}{\partial\lambda} E \approx -\frac{m}{4}$$

which is in agreement with the result (13) .

3 Rotating string models

In our models we consider a point-like string on the AdS space i.e.

$$x = x(\tau)\ ,\ z = z(\tau) \tag{14}$$

and a general spinning string in the X_5 space [35] described by

$$\phi_1 = \omega_1 \tau\ ,\ \phi_2 = \omega_2 \tau\ ,\ \theta_1 = \theta_1(\sigma)\ ,\ \theta_2 = \theta_2(\sigma)\ ,\ \psi = \nu\tau. \tag{15}$$

Inserting the ansatz (14),(15) into (3) we obtain the action $S[X, \Phi] = -\frac{\sqrt{\lambda}}{4\pi} \int (\mathcal{L}_X + \mathcal{L}_A)\, d\sigma d\tau$ with

$$\mathcal{L}_A = \frac{\overrightarrow{x}'(\tau)^2 + z'(\tau)^2}{z(\tau)^2} \tag{16}$$

being the Lagrangian for the AdS part and

$$\begin{aligned}\mathcal{L}_X = b\big(&-\nu^2 - 2\nu\omega_1 \cos(\theta_1(\sigma)) - 2\nu\omega_2 \cos(\theta_2(\sigma)) \\ &-2\omega_1\omega_2 \cos(\theta_1(\sigma))\cos(\theta_2(\sigma))\big) - \omega_1^2 \left(b\cos^2(\theta_1(\sigma)) + a\sin^2(\theta_1(\sigma))\right) \\ &- \omega_2^2 \left(b\cos^2(\theta_2(\sigma)) + a\sin^2(\theta_2(\sigma))\right) + a\left(\theta_1'(\sigma)^2 + \theta_2'(\sigma)^2\right).\end{aligned} \tag{17}$$

specifying the X_5 part. Similarly we find the effective action $\tilde{S}[X, s]$. According to (10) it is enough to write down the "subtracted" action $\bar{S}[X, s] = -\frac{\sqrt{\lambda}}{4\pi} \int \left(\bar{\mathcal{L}}_X + \bar{\mathcal{L}}_A\right) d\sigma d\tau$, substituting the ansatz (14),(15) to (10). The Lagrangians we obtain are of the following form

$$\begin{aligned}\bar{\mathcal{L}}_A =& 0 \\ \bar{\mathcal{L}}_X =& -2b\left(\nu^2 + 2\nu\omega_1 \cos(\theta_1(\sigma)) + 2\nu\omega_2 \cos(\theta_2(\sigma))\right. \\ &\left.+\omega_1^2 \cos^2(\theta_1(\sigma)) + \omega_2^2 \cos^2(\theta_2(\sigma)) + 2\omega_1\omega_2 \cos(\theta_1(\sigma))\cos(\theta_2(\sigma))\right) \\ &- 2a\left(\omega_1^2 \sin^2(\theta_1(\sigma)) + \omega_2^2 \sin^2(\theta_2(\sigma))\right).\end{aligned} \tag{18}$$

There are three conserved charges $J_{\phi_1}, J_{\phi_2}, J_\psi$ corresponding to the symmetries of the X_5 space with their currents $P_{\phi_1}, P_{\phi_2}, P_\psi$ satisfying $J_I = -\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} P_I d\sigma$, where $I \in \{A, B, R\}$. Evaluating the currents P_I for the Lagrangian $\mathcal{L} = \mathcal{L}_A + \mathcal{L}_X$ (16),(17) we obtain

$$\begin{aligned}P_A = P_{\phi_1} =& \frac{\partial \mathcal{L}}{\partial(\partial_\tau \phi_1)} = -2\left(\left(b\cos^2(\theta_1(\sigma)) + a\sin^2(\theta_1(\sigma))\right)\phi_1'(\tau)\right) \\ &- 2\left(b\cos(\theta_1(\sigma))\left(\cos(\theta_2(\sigma))\phi_2'(\tau) + \psi'(\tau)\right)\right) \\ P_B = P_{\phi_2} =& \frac{\partial \mathcal{L}}{\partial(\partial_\tau \phi_2)} = -2\left(b\cos(\theta_1(\sigma))\cos(\theta_2(\sigma))\phi_1'(\tau)\right) \\ &- 2\left(\left(b\cos^2(\theta_1(\sigma)) + a\sin^2(\theta_1(\sigma))\right)\phi_2'(\tau) + b\cos(\theta_2(\sigma))\psi'(\tau)\right) \\ P_R = P_\psi =& \frac{\partial \mathcal{L}}{\partial(\partial_\tau \psi)} = -2b\left(\cos(\theta_1(\sigma))\phi_1'(\tau) + \cos(\theta_2(\sigma))\phi_2'(\tau) + \psi'(\tau)\right).\end{aligned} \tag{19}$$

Here and in the following we use *prime* for the derivation when there is one variable only. The Virasoro constraints can be written in a concise form

$$vir_1 = vir_{1A} + vir_{1X} = T_{\tau\tau} + T_{\sigma\sigma}\ ,\ vir_2 = vir_{2A} + vir_{2X} = T_{\tau\sigma}$$

where T_{ab} is the world-sheet energy-momentum tensor. With the ansatz (14) their explicit form for the AdS part is

$$vir_{1A} = \frac{x'(\tau)^2 + z'(\tau)^2}{z(\tau)^2}\ ,\ vir_{2A} = 0.$$

Similarly using the ansatz (15) we obtain the X_5 part as

$$
\begin{aligned}
vir_{1X} = b \big(\nu^2 + 2\nu\omega_1 \cos(\theta_1(\sigma)) + \omega_1^2 \cos^2(\theta_1(\sigma)) + 2\nu\omega_2 \cos(\theta_2(\sigma)) \\
+2\omega_1\omega_2 \cos(\theta_1(\sigma)) \cos(\theta_2(\sigma)) + \omega_2^2 \cos^2(\theta_2(\sigma))\big) \\
+ a \left(\omega_1^2 \sin^2(\theta_1(\sigma)) + \omega_2^2 \sin^2(\theta_2(\sigma)) + \theta_1'(\sigma)^2 + \theta_2'(\sigma)^2\right)
\end{aligned}
$$

and

$$vir_{2X} = 0.$$

In the following part we analyze particular string solutions.

3.1 single spin case

The single spin rotating string extended in the θ_1 direction and rotating in the ϕ_1 is given by the following consistent truncation of (15)

$$\theta_1(\sigma) = \theta(\sigma) \ , \ \theta_2(\sigma) = 0 \, , \, \omega_1 = \omega \, , \, \omega_2 = 0 \, , \, \nu = 0. \tag{20}$$

Substituting the ansatz (20) to (16),(17),(18) we find the following Lagrangians

$$
\begin{aligned}
\mathcal{L}_1 =& -\frac{x'(\tau)^2 + z'(\tau)^2}{z(\tau)^2} - \omega^2 \left(b \cos^2(\theta(\sigma)) + a \sin^2(\theta(\sigma))\right) + a\theta'(\sigma)^2 \\
\bar{\mathcal{L}}_1 =& -2\omega^2 \left(b \cos^2(\theta(\sigma)) + a \sin^2(\theta(\sigma))\right).
\end{aligned} \tag{21}
$$

They result in the action

$$\tilde{S}_1\left[\bar{X}, s\right] = -\frac{\sqrt{\lambda}}{4\pi} \left(-2\pi\kappa^2 s + s \int_0^{2\pi} \left(\omega^2 \left(b \cos^2(\theta(\sigma)) + a \sin^2(\theta(\sigma))\right) + a\theta'(\sigma)^2\right) d\sigma\right) \tag{22}$$

where we already put the solution for the AdS part (5) . The subscript 1 indicates the single spin case. Rewriting the action (22) in a convenient form and substituting κ from (7) we obtain

$$\tilde{S}_1\left[\bar{X}, s\right] = -\frac{\sqrt{\lambda}}{4\pi} \left(-2\pi\kappa^2 s + sI_C\right) = -\frac{\sqrt{\lambda}}{4\pi} \left(-\frac{8\pi \log^2\left(\frac{x_f}{\epsilon}\right)}{s} + sI_C\right) \tag{23}$$

where I_C is given by

$$I_C = \int_0^{2\pi} \left(\omega^2 \left(b \cos^2(\theta(\sigma)) + a \sin^2(\theta(\sigma))\right) + a\theta'(\sigma)^2\right) d\sigma. \tag{24}$$

The conserved charges J_I for the single spin Lagrangian (21) takes the following form:

$$
\begin{aligned}
J_A =& \frac{\sqrt{\lambda}}{2\pi} \omega \int_0^{2\pi} d\sigma \left(b \cos^2(\theta(\sigma)) + a \sin^2(\theta(\sigma))\right) \\
J_B =& \frac{\sqrt{\lambda}}{2\pi} \omega b \int_0^{2\pi} d\sigma \cos(\theta(\sigma)) \\
J_R =& \frac{\sqrt{\lambda}}{2\pi} \omega b \int_0^{2\pi} d\sigma \cos(\theta(\sigma)).
\end{aligned} \tag{25}
$$

Next we proceed with calculation of the 2-point and the 3-point function keeping the $\theta(\sigma)$ unsolved for the moment. We calculate the 2-point function using the saddle point approximation. Since the integral (24) is independent of the modular parameter s, the saddle point $\bar{s}$ of the action (23) is of the form

$$\bar{s} = -\frac{2\imath\sqrt{2\pi} \log\left(\frac{x_f}{\epsilon}\right)}{\sqrt{I_C}}. \tag{26}$$

The action $\tilde{S}_1$ at the above saddle point becomes

$$\tilde{S}_1\left[\bar{X},\bar{s}\right]=\imath\sqrt{\frac{2}{\pi}}\sqrt{\lambda}\sqrt{I_C}\log\left(\frac{x_f}{\epsilon}\right),$$

and results in the 2-point function of the form

$$\langle\mathcal{O}\left(0\right),\mathcal{O}\left(x_f\right)\rangle\approx e^{i\tilde{S}_1\left[\bar{X},\bar{s}\right]}=\left(\frac{x_f}{\epsilon}\right)^{-\sqrt{\frac{2}{\pi}}\sqrt{\lambda}\sqrt{I_C}}. \tag{27}$$

In order to obtain the three-point function one has to calculate the interaction term I_ϕ as indicated in (11) (see [18] for details). Inserting the Lagrangian (21) to the relation (12), the integrand i_ϕ for the single spin string becomes

$$i_{\phi_1}=\frac{3\imath x_f^4\left(\frac{x_f}{\epsilon}\right)^{\frac{8\tau}{s}}}{2\pi^2 s^2\left(y^2+\left(x_f-y\right)^2\left(\frac{x_f}{\epsilon}\right)^{\frac{4\tau}{s}}\right)^4}\left(s^2\omega^2\left(-a-b+\left(a-b\right)\cos\left(2\theta\left(\sigma\right)\right)\right)\right.$$
$$\left.-8\log^2\left(\frac{x_f}{\epsilon}\right)+2as^2\theta'^2\left(\sigma\right)\right).$$

Using a small ϵ approximation for the interaction term I_ϕ we find

$$I_{\phi_1}\approx\left(\int_0^{2\pi}\left(s^2\omega^2\left(a+b-\left(a-b\right)\cos\left(2\theta\left(\sigma\right)\right)\right)+8\log^2\left(\frac{x_f}{\epsilon}\right)-2as^2\theta'^2\left(\sigma\right)\right)d\sigma\right)$$
$$\frac{\imath x_f^4\sqrt{\lambda}}{64\pi^3 s\left(x_f-y\right)^4 y^4\log\left(\frac{x_f}{\epsilon}\right)},$$

which at the saddle point (26) turns to

$$I_{\phi_1}\approx\frac{\imath x_f^4\sqrt{\lambda}}{32\pi^3\left(x_f-y\right)^4 y^4\kappa}\int_0^{2\pi}\left(2\kappa^2+\omega^2\left(a+b-\left(a-b\right)\cos\left(2\theta\left(\sigma\right)\right)\right)-2a\theta'^2\left(\sigma\right)\right)d\sigma. \tag{28}$$

Using the Virasoro constraint (30) and the J_A charge (25) the interaction term (28) becomes

$$I_{\phi_1}\approx\frac{\imath\left(\kappa^2\sqrt{\lambda}+\omega J_A\right)}{4\pi^2\kappa}\frac{x_f^4}{\left(x_f-y\right)^4 y^4}.$$

According to (11) we find the following three-point function

$$\langle\mathcal{O}_A\left(0\right)\mathcal{O}_A\left(x_f\right)\mathcal{L}\left(y\right)\rangle\approx\frac{\imath\left(\kappa^2\sqrt{\lambda}+\omega J_A\right)}{4\pi^2\kappa}\frac{1}{x_f^{2\Delta_A-4}y^4\left(x_f-y\right)^4}. \tag{29}$$

The two (27) and three (29) point function we obtained above include the yet unsolved $\theta\left(\sigma\right)$ in a more or less hidden form. In order to solve for $\theta\left(\sigma\right)$ we use the first Virasoro constraint

$$\kappa^2+b\omega^2+\left(a-b\right)\omega^2\sin^2\left(\theta\left(\sigma\right)\right)+a\theta'^2\left(\sigma\right)=0, \tag{30}$$

where all parameters except of κ are real, so we demand $\kappa^2+b\omega^2<0$. Introducing the following parameters

$$c=\imath\frac{\sqrt{\kappa^2+b\omega^2}}{\sqrt{a}},\ k=-\frac{\left(a-b\right)\omega^2}{\kappa^2+b\omega^2} \tag{31}$$

we get a convenient form of the equation (30)

$$\frac{\partial\theta\left(\sigma\right)}{\partial\sigma}=c\sqrt{1-k\sin^2\left(\theta\left(\sigma\right)\right)}. \tag{32}$$

It is necessary at this point to distinguish between two cases depending on k :

1. $k > 1$ where $\theta(\sigma)$ reaches a turning point and thus describes a folded string.
2. $k < 1$ where the function $\theta(\sigma)$ is monotonous and thus describes a circular string. Such a string winds around the great circle of the S^2.

We analyze these cases separately below. For completeness we note that both cases are constrained by positive value of the parameter k.

Folded string We start our analysis with the folded string case. In order to bring the equation (32) with $k > 1$ into a convenient form we use the transformation

$$k \sin^2 \theta(\sigma) = \sin^2 X(\sigma), \tag{33}$$

leading to the following equation

$$\frac{\partial X(\sigma)}{\partial \sigma} = c\sqrt{k}\sqrt{1 - \frac{1}{k}\sin^2 (X(\sigma))}.$$

The solution to this differential equation subject to the condition $X(0) = 0$, is expressed in terms of the Jacobi Amplitude ($am(z, m) = \mathbb{F}^{-1}(z, m)$) as

$$X(\sigma) = am\left(c\sqrt{k}\sigma, \frac{1}{k}\right). \tag{34}$$

The closed string periodicity condition

$$X(0) = X(2\pi) + 2\pi \tag{35}$$

demands the constant c to be

$$c = -\frac{2\mathbb{K}\left(\frac{1}{k}\right)}{\sqrt{k}\pi}, \tag{36}$$

bringing up the final form of $X(\sigma)$

$$X(\sigma) = am\left(-\frac{2\mathbb{K}\left(\frac{1}{k}\right)}{\pi}\sigma, \frac{1}{k}\right). \tag{37}$$

The expression (36) with the definition of parameters (31) results in a relation between κ and ω of the form

$$\mathbb{K}\left(\frac{1}{k}\right) = \mathbb{K}\left(-\frac{\kappa^2 + b\omega^2}{(a-b)\,\omega^2}\right) = \frac{\pi}{2}\omega\frac{\sqrt{a-b}}{\sqrt{a}}. \tag{38}$$

Inserting the solution (37) and the substitution (33) into the single-spin relations for conserved charges (25) we obtain

$$\begin{aligned} J_B = J_R &= \frac{\sqrt{\lambda} b\pi\omega}{2\mathbb{K}\left(\frac{1}{k}\right)} = \frac{\sqrt{a}b\sqrt{\lambda}}{\sqrt{a-b}} \\ J_A &= \frac{\sqrt{\lambda}\omega\left(a\mathbb{K}\left(\frac{1}{k}\right) - (a-b)\,\mathbb{E}\left(\frac{1}{k}\right)\right)}{\mathbb{K}\left(\frac{1}{k}\right)} = a\sqrt{\lambda}\omega - \frac{2}{\pi}\sqrt{\lambda}\sqrt{a}\sqrt{a-b}\mathbb{E}\left(\frac{1}{k}\right) \end{aligned} \tag{39}$$

where in the last equality we already used the relation (38). Expressing κ from (38) we calculate the energy

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} -\imath\kappa d\sigma = \sqrt{\lambda}\omega\sqrt{b + (a-b)\,\mathbb{K}^{-1}\left(\frac{\pi\omega\sqrt{a-b}}{2\sqrt{a}}\right)}. \tag{40}$$

We note that in order to get the dispersion relation from (40) one needs to substitute ω from the J_A charge (39). It is however a transcendental equation because k depends on ω and κ (31), hence there is no algebraic expression for the dispersion relation.

Let us return to the two (27) and the three (29) -point function calculated for the single spin string. In order to complete the two-point function calculation we need to express the square root $\sqrt{I_C}$ of the integral (24) in terms of constants and conserved charges. It is the same as having the dispersion relation, which however can not be solved algebraically in this case. A similar argument applies for the three-point function (29), where in order to express it with charges and constants only, one needs to find the relation for ω/κ. It again involves transcendental equation and therefore can not be obtained explicitly.

In order to compare our result with the expectation from the renormalization flow equation [18] we need to calculate the derivative $\frac{\partial}{\partial\lambda}E$. Since the energy (40) is given implicitly we proceed as follows :

$$2\pi^2 a_{\mathcal{L}AA} = -\lambda\frac{\partial}{\partial\lambda}E = \lambda\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda}\imath\kappa\right) = \imath\frac{\sqrt{\lambda}}{2}\left(\kappa + 2\lambda\frac{\partial\kappa}{\partial\lambda}\right) \tag{41}$$

where E is the energy and $a_{\mathcal{L}AA}$ the coupling. Using (31) we express the derivative $\frac{\partial\kappa}{\partial\lambda}$ in terms of the other parameters

$$\frac{\partial\kappa}{\partial\lambda} = \frac{\kappa^2 + b\omega^2}{2\,(a-b)\,\kappa\omega^2}\frac{\partial k}{\partial\lambda} + \frac{\kappa}{\omega}\frac{\partial\omega}{\partial\lambda}, \tag{42}$$

and from the relation (38) we obtain $\frac{\partial k}{\partial\lambda}$ of the form

$$\frac{\partial k}{\partial\lambda} = -\frac{\sqrt{a-b}\,(k-1)\,k\pi\frac{\partial\omega}{\partial\lambda}}{\sqrt{a}k\mathbb{E}\left(\frac{1}{k}\right) - (k-1)\,\mathbb{K}\left(\frac{1}{k}\right)}. \tag{43}$$

Having (43) the last derivative we need to express is $\frac{\partial\omega}{\partial\lambda}$. Differentiating the J_A charge (39) we find

$$\frac{\partial\omega}{\partial\lambda} = \frac{\sqrt{a}\left(\sqrt{a}\pi\omega - 2\sqrt{a-b}\mathbb{E}\left(\frac{1}{k}\right)\right)\left(k\mathbb{E}\left(\frac{1}{k}\right) - (k-1)\,\mathbb{K}\left(\frac{1}{k}\right)\right)}{2\pi\lambda\left(-\,(a+b\,(k-1))\,\mathbb{E}\left(\frac{1}{k}\right) + b\,(k-1)\,\mathbb{K}\left(\frac{1}{k}\right)\right)}. \tag{44}$$

Finally substituting (43) and (44) into (42) and using the relations (39),(38) we find the same result as from the saddle point approximation, i.e.

$$a_{\mathcal{L}AA} = \imath\frac{\kappa^2\sqrt{\lambda} + \omega J_A}{4\pi^2\kappa}.$$

circular string case As indicated in the analysis of the Virasoro constraint (32) the circular string case is realized when $k<1$. For this range of the parameter k we get a solution to θ in the form

$$\theta(\sigma) = am\,(c\sigma, k)\,. \tag{45}$$

Due to the periodicity condition, which in this case includes winding around the great circle of S^2 we have

$$am\,(2\pi c, k) = am\,(0,k) + 2\pi, \tag{46}$$

which brings up the following relation between κ and ω

$$\mathbb{K}\,(k) = \mathbb{K}\left(-\frac{(a-b)\,\omega^2}{\kappa^2 + b\omega^2}\right) = \frac{\pi}{2}c = \imath\frac{\pi}{2}\frac{\sqrt{\kappa^2+b\omega^2}}{\sqrt{a}}. \tag{47}$$

The solution for $\theta(\sigma)$ is then

$$\theta(\sigma) = am\left(\frac{2\mathbb{K}\,(k)}{\pi}\sigma, k\right). \tag{48}$$

In order to find an explicit solution to the conserved charges (25) we insert the solution (48) to the relations for charges (25), and using the relation (47) we obtain

$$\begin{aligned} J_B &= J_R = 0 \\ J_A &= -\frac{\kappa^2\sqrt{\lambda}}{\omega} - \frac{2\imath\sqrt{a}\sqrt{\lambda}\sqrt{\kappa^2+b\omega^2}\,\mathbb{E}\left(k\right)}{\pi\omega}. \end{aligned} \tag{49}$$

An explicit relation for energy in the circular string case is more complicated, because the relation (47) includes κ on the right hand side. It is therefore not possible to express κ from (47) in an algebraic way, therefore the energy

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} -\imath\kappa d\sigma = -\sqrt{\lambda}\imath\kappa, \tag{50}$$

is subjected to the relation for κ (47).

Similar to the folded case having an explicit relation for the 2-point function means having the dispersion relation. In order to find it one has to solve a set of equations $\{(50),(47),(49)\}$, which again can not be done algebraically. The same argumentation applies to the three-point function (29) where one has to find the ratio ω/κ , which can not be done explicitly. The difference in the 2-point and the 3-point function between the folded and the circular single spin string is in the relation between κ and ω (47) and the J_A charge (49).

We proceed comparing the 3-point function to the expectation from the renormalization flow, starting with the expression

$$2\pi^2 a_{\mathcal{L}AA} = -\lambda\frac{\partial}{\partial\lambda}E = \lambda\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda}\imath\kappa\right) = \imath\frac{\sqrt{\lambda}}{2}\left(\kappa + 2\lambda\frac{\partial\kappa}{\partial\lambda}\right). \tag{51}$$

In order to compute $\frac{\partial\kappa}{\partial\lambda}$ we differentiate the parameter k (31) and the relation (47) obtaining the following equations

$$\begin{aligned} \frac{\partial k}{\partial\lambda}\left(\kappa^2+b\omega^2\right)^2 &= 2\left(a-b\right)\kappa\omega m\left(\omega\frac{\partial\kappa}{\partial\lambda} - \kappa\frac{\partial\omega}{\partial\lambda}\right) \\ \frac{\frac{\partial k}{\partial\lambda}\left(\mathbb{E}\left(k\right)-\left(1-k\right)\mathbb{K}\left(k\right)\right)}{2\left(1-k\right)k} &= \frac{\imath\pi\left(\kappa\frac{\partial\kappa}{\partial\lambda}+b\omega\frac{\partial\omega}{\partial\lambda}\right)}{2\sqrt{a}\sqrt{\kappa^2+b\omega^2}}. \end{aligned}$$

Solving them for $\frac{\partial\kappa}{\partial\lambda}$ and $\frac{\partial k}{\partial\lambda}$ we find the relations

$$\frac{\partial\kappa}{\partial\lambda} = \left(\frac{\kappa}{\omega} - \frac{\imath\pi\left(\kappa^2+a\omega^2\right)\sqrt{\kappa^2+b\omega^2}}{2\kappa\omega\sqrt{a}\,\mathbb{E}\left(k\right)}\right)\frac{\partial\omega}{\partial\lambda} \tag{52}$$

$$\frac{\partial k}{\partial\lambda} = -\frac{\imath\pi\omega\left(a-b\right)\left(\kappa^2+a\omega^2\right)}{\sqrt{a}\left(\kappa^2+b\omega^2\right)^{3/2}\mathbb{E}\left(k\right)}\frac{\partial\omega}{\partial\lambda}, \tag{53}$$

which substituted to $\frac{\partial J_A}{\partial\lambda} = 0$ results in

$$\frac{\partial\omega}{\partial\lambda} = \frac{\omega\mathbb{E}\left(k\right) - \sqrt{a}\pi\kappa^2 - 2\imath a\sqrt{k^2+b\omega^2}\,\mathbb{E}\left(k\right)}{\pi\lambda\left(2\sqrt{a}\kappa^2\mathbb{E}\left(k\right) - \imath\pi\left(\kappa^2+a\omega^2\right)\sqrt{k^2+b\omega^2}\right)}. \tag{54}$$

Finally substituting (54) and (52) to (51) and using the relation (47) we end up with

$$a_{\mathcal{L}AA} = \frac{\imath\left(\kappa\sqrt{\lambda}+\frac{\omega}{\kappa}J_A\right)}{4\pi^2} = \frac{-E+\imath\frac{\omega}{\kappa}J_A}{4\pi^2}$$

which coincides with the result (29).

3.2 Two spin folded string

In the following case we analyze a string extended in the θ_1 and rotating in the Φ_1 and Ψ direction, described by a consistent truncation of (15) of the form

$$\theta_1(\sigma) = \theta(\sigma)\ ,\ \theta_2(\sigma) = 0\,,\ \omega_1 = \omega\,,\ \omega_2 = 0\,,\ \nu \neq 0 \implies \Psi = \nu\tau.$$

For the above ansatz the Lagrangians $\mathcal{L}$ (16),(17) and $\bar{\mathcal{L}}$ (18) becomes

$$\mathcal{L}_2 = -\frac{x'(\tau)^2 + z'(\tau)^2}{z(\tau)^2} - \omega^2\left(b\cos^2(\theta(\sigma)) + a\sin^2(\theta(\sigma))\right) + a\theta'(\sigma)^2 - b\nu^2 - 2b\nu\omega\cos(\theta(\sigma)) \tag{55}$$

$$\bar{\mathcal{L}}_2 = -2\left[\omega^2\left(b\cos^2(\theta(\sigma)) + a\sin^2(\theta(\sigma))\right) + b\nu^2 + 2b\nu\omega\cos(\theta(\sigma))\right]. \tag{56}$$

We use the subscript 2 to distinguish the two spin case when necessary. The action $\tilde{S}_2$ with the solution for the AdS part (5) already inserted, takes the form

$$\tilde{S}_2\left[\bar{X}, s\right] = -\frac{\sqrt{\lambda}}{4\pi}\left(-2\pi s\left(\kappa^2 - b\nu^2\right) + s\int_0^{2\pi}\left(\omega^2\left(b\cos^2(\theta(\sigma)) + a\sin^2(\theta(\sigma))\right)\right.\right.$$
$$\left.\left. +2b\nu\omega\cos(\theta(\sigma)) + a\theta'(\sigma)^2\right)d\sigma\right). \tag{57}$$

In order to find the solution for $\theta(\sigma)$ we use the first Virasoro constraint obtaining

$$-a\theta'^2(\sigma) = \kappa^2 + b\nu^2 + a\omega^2 + 2b\nu\omega\cos(\theta(\sigma)) - (a-b)\,\omega^2\cos^2(\theta(\sigma)). \tag{58}$$

The above equation includes a term linear in $\cos(\theta(\sigma))$, therefore it is useful to rewrite (58) in a more convenient form

$$\theta'^2(\sigma) = \frac{a-b}{a}\omega^2\left(\cos(\theta(\sigma)) - \alpha\right)\left(\cos(\theta(\sigma)) - \beta\right), \tag{59}$$

where α and β are roots of the right hand side of (59) regarded as a polynomial in $\cos(\theta)$. A folded string we analyze, extends between a minimal and a maximal angle $\theta \in \langle\theta_{min}, \theta_{max}\rangle \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which in terms of $\cos(\theta)$ means $\cos(\theta) \in (0, 1\rangle$. First we note that without loss of generality we set $\theta(0) = 0$ and that a folded string requires at least one of the roots α, β to be the turning point. Regarding the right hand side of (59) as a quadratic polynomial in $\cos(\theta)$, we find that in order to reach a turning point i.e. to have a *non-negative* region we must set $\alpha, \beta \in (-1, 1)$. Due to $\nu \neq 0$ the roots α and β are different, so we choose $\beta > \alpha$. In order to have a turning point we must set $0 < \beta < 1$ and because of the consistency of the folded string we assume also $-1 < \alpha < 0$. With all these constraints we find that the turning point is at $\cos(\theta_{turn}) = \beta$ and that the folded string extends between $\pm\theta_{turn}$ i.e. $\theta \in (-\theta_{turn}, \theta_{turn})$. The roots α and β have the following form

$$\alpha = \frac{b\nu - \sqrt{(a-b)\,\kappa^2 + ab\nu^2 + a\,(a-b)\,\omega^2}}{(a-b)\,\omega}$$
$$\beta = \frac{b\nu + \sqrt{(a-b)\,\kappa^2 + ab\nu^2 + a\,(a-b)\,\omega^2}}{(a-b)\,\omega}. \tag{60}$$

Solving the equation (59) for $\theta(\sigma)$ we get the following expression

$$\sqrt{\frac{a-b}{a}}\omega\sigma = \int_0^{\theta}\frac{\mathbf{d}\theta}{\sqrt{\left(\cos(\theta(\sigma)) - \alpha\right)\left(\cos(\theta(\sigma)) - \beta\right)}} = I_1(\theta) - I_1(0) \tag{61}$$

giving the solution to $\theta(\sigma)$ only implicitly. This is because the integral $I_1(\theta)$ (A.3) is up to a multiplicative factor the elliptic integral of the first kind. The folded string periodicity condition gives the following relation

$$2\pi\sqrt{\frac{a-b}{a}}\omega = 4\int_0^{\theta_{turn}}\frac{\mathbf{d}\theta}{\sqrt{\left(\cos(\theta(\sigma)) - \alpha\right)\left(\cos(\theta(\sigma)) - \beta\right)}} = 4\left(I_1(acrcos(\beta)) - I_1(0)\right). \tag{62}$$

Substituting for $I_1(\theta)$ from (A.3) the relation (62) becomes

$$\pi\omega = 4\sqrt{\frac{a}{a-b}}\frac{\mathbb{K}(k)}{\sqrt{(1-\alpha)(1+\beta)}}, \tag{63}$$

where k is defined as

$$k = \frac{(1+\alpha)(1-\beta)}{(1-\alpha)(1+\beta)}. \tag{64}$$

Inserting the two spin Lagrangian (55) in the relations for currents (19), one finds the following expressions for conserved charges

$$\begin{aligned} J_A =& \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\left[\omega\left(b\cos^2(\theta(\sigma)) + a\sin^2(\theta(\sigma))\right) + b\nu\cos(\theta(\sigma))\right] \\ J_R = J_B =& \frac{\sqrt{\lambda}}{2\pi}b\int_0^{2\pi} d\sigma\left(\nu + \omega\cos(\theta(\sigma))\right). \end{aligned} \tag{65}$$

The solution to $\theta(\sigma)$ (61) is defined implicitly, so we can not insert it directly to the relations (65). It is therefore better to make a substitution $\sigma \to \theta(\sigma)$ in the above integrals based on (59), which for an arbitrary function $f(\sigma)$ becomes

$$\int_0^{\frac{\pi}{2}} f(\sigma)\,d\sigma \to \int_0^{\theta_{turn}} f(\theta(\sigma))\frac{d\theta}{\theta'} = \int_0^{arccos(\beta)} \frac{f(\theta(\sigma))\,d\theta}{\sqrt{(\cos(\theta(\sigma))-\alpha)(\cos(\theta(\sigma))-\beta)}}.$$

There are three integrals entering the calculation, which we denote as I_1, I_2, I_3 with the following definition:

$$\begin{aligned} I_1 &= \int_0^{arccos(\beta)} \frac{1}{\sqrt{(\cos(\theta)-\alpha)(\cos(\theta)-\beta)}}d\theta \\ I_2 &= \int_0^{arccos(\beta)} \frac{\cos(\theta)}{\sqrt{(\cos(\theta)-\alpha)(\cos(\theta)-\beta)}}d\theta \\ I_3 &= \int_0^{arccos(\beta)} \sqrt{(\cos(\theta)-\alpha)(\cos(\theta)-\beta)}d\theta. \end{aligned} \tag{66}$$

The reader can find their expression in the appendix (A.4). Rewriting relations for the charges (65) using the integrals (66) we obtain

$$\begin{aligned} J_A &= -\frac{\sqrt{\lambda}}{\pi\omega^2}\sqrt{\frac{a}{a-b}}\left(\left(\kappa^2 + b\nu^2\right)I_1 + \omega\left((a-b)\,\omega I_3 + b\nu I_2\right)\right) \\ J_B &= \frac{\sqrt{\lambda}}{\pi\omega}2b\sqrt{\frac{a}{a-b}}\left(\nu I_1 + \omega I_2\right). \end{aligned} \tag{67}$$

Substituting the integrals I_i from (A.4), the charges (67) becomes

$$\begin{aligned} J_A =& -\frac{\sqrt{\lambda}}{\pi\omega^2}\sqrt{\frac{a}{a-b}}\frac{2}{\sqrt{(1-\alpha)(1+\beta)}}\left((a-b)(1-\alpha)(1+\beta)\,\omega^2\mathbb{E}(k)\right. \\ &+ 2\left(\kappa^2 + b\nu^2 - b\nu\omega + (a-b)\,\alpha(1+\beta)\,\omega^2\right)\mathbb{K}(k) \\ &\left.+2\omega\left(2b\nu - (a-b)(\alpha+\beta)\,\omega\right)\mathbb{PI}(n,k)\right) \\ J_B =& \frac{\sqrt{\lambda}}{\pi\omega}2b\sqrt{\frac{a}{a-b}}\frac{2}{\sqrt{(1-\alpha)(1+\beta)}}\left((\nu-\omega)\,\mathbb{K}(k) + 2\omega\mathbb{PI}(n,k)\right). \end{aligned} \tag{68}$$

In order to calculate the energy

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} -\imath\kappa d\sigma = -\sqrt{\lambda}\imath\kappa, \tag{69}$$

one needs to express κ from the relation (63). It is again a complicated transcendental equation, which therefore determines κ only implicitly. The dispersion relation given via the system of equations $\{(69),(63),(68)\}$ can not be solved in an algebraic way either.

The calculation of the 2-point function follows the same steps as in cases discussed before. Using the integral

$$I_{C2} = \int_0^{2\pi} \left(\omega^2 \left(b\cos^2(\theta(\sigma)) + a\sin^2(\theta(\sigma))\right) + 2b\omega\nu\cos(\theta(\sigma)) + a\theta'(\sigma)^2\right) d\sigma,$$

the solution (5) and the relation for κ (7) we rewrite the action (57) in the following concise form

$$\tilde{S}_2\left[\bar{X}, s\right] = -\frac{\sqrt{\lambda}}{4\pi}\left(-\frac{8\pi\log^2\left(\frac{x_f}{\epsilon}\right)}{s} + s\left(2\pi b\nu^2 + I_{C2}\right)\right). \tag{70}$$

The above action has the following saddle point

$$\bar{s} = -\frac{2\imath\sqrt{2\pi}\log\left(\frac{x_f}{\epsilon}\right)}{\sqrt{2\pi b\nu^2 + I_{C2}}}. \tag{71}$$

Evaluating the action (70) at the saddle point (71) we find

$$\tilde{S}_2\left[\bar{X}, \bar{s}\right] = \imath\sqrt{\frac{2}{\pi}}\sqrt{\lambda}\sqrt{2\pi b\nu^2 + I_{C2}}\log\left(\frac{x_f}{\epsilon}\right),$$

which results in the following 2-point function

$$\langle\mathcal{O}(0), \mathcal{O}(x_f)\rangle \approx e^{i\tilde{S}_2\left[\bar{X},\bar{s}\right]} = \left(\frac{x_f}{\epsilon}\right)^{-\sqrt{\frac{2}{\pi}}\sqrt{\lambda}\sqrt{2\pi b\nu^2 + I_{C2}}}.$$

The integrand i_ϕ (12) of the interaction term becomes in the two spin case

$$i_{\phi_2} = -\frac{3\imath x_f^4\left(\frac{x_f}{\epsilon}\right)^{\frac{8\pi}{s}}}{2\pi^2 s^2\left(y^2 + (x_f - y)^2\left(\frac{x_f}{\epsilon}\right)^{\frac{4\pi}{s}}\right)^4}\left(s^2\omega^2\left(a + b - (a-b)\cos(2\theta(\sigma))\right)\right.$$
$$\left. + s^2\left(2b\nu^2 + 4b\nu\omega\cos(\theta(\sigma)) - 2a\theta'^2(\sigma)\right) + 8\log^2\left(\frac{x_f}{\epsilon}\right)\right).$$

A small ϵ approximation of the interaction term results in the following expression

$$I_{\phi_2} \approx \left(\int_0^{2\pi}\left(s^2\omega^2\left(a - (a-b)\cos^2(\theta(\sigma))\right) + 4\log^2\left(\frac{x_f}{\epsilon}\right)\right.\right.$$
$$\left.\left. + s^2\left(b\nu^2 + 2b\nu\omega\cos(\theta(\sigma)) - a\theta'^2(\sigma)\right)\right)\sigma\right)\frac{\imath x_f^4\sqrt{\lambda}}{32\pi^3 s\,(x_f - y)^4 y^4\log\left(\frac{x_f}{\epsilon}\right)}.$$

Inserting the saddle point (71) and using (58) we obtain

$$I_{\phi_2} \approx \frac{\imath x_f^4\sqrt{\lambda}}{8\pi^3(x_f - y)^4 y^4\kappa}\int_0^{2\pi}\left(\kappa^2 + b\nu^2 + 2b\nu\omega\cos(\theta(\sigma)) + \omega^2\left(a - (a-b)\cos^2(\theta(\sigma))\right)\right)d\sigma. \tag{72}$$

In order to rewrite the interaction term (72) in terms of conserved charges we use the relations (65) and obtain

$$I_{\phi_2} \approx \frac{\imath\left(\kappa^2\sqrt{\lambda} + \omega J_A + \nu J_B\right)}{4\pi^2\kappa}\frac{x_f^4}{(x_f - y)^4 y^4},$$

which results in the three-point function of the form

$$\langle \mathcal{O}_A(0)\,\mathcal{O}_A(x_f)\,\mathcal{L}(y)\rangle \approx \frac{\imath\left(\kappa^2\sqrt{\lambda}+\omega J_A+\nu J_B\right)}{4\pi^2\kappa}\frac{1}{x_f^{2\Delta_A-4}y^4\left(x_f-y\right)^4}. \tag{73}$$

In the last part of this section we compare our result of the three-point function (73) to the expectation from the renormalization flow equation. Since energy is given by an implicit relation we start with the following expression

$$2\pi^2 a_{\mathcal{L}AA} = -\lambda\frac{\partial}{\partial\lambda}E = \lambda\frac{\partial}{\partial\lambda}\left(\sqrt{\lambda}\imath\kappa\right) = \imath\frac{\sqrt{\lambda}}{2}\left(\kappa+2\lambda\frac{\partial\kappa}{\partial\lambda}\right). \tag{74}$$

In order to find $\frac{\partial\kappa}{\partial\lambda}$ we use the following equations

$$\frac{\partial J_A}{\partial\lambda} = 0\,,\ \frac{\partial J_B}{\partial\lambda} = 0 \tag{75}$$

$$\pi\frac{\partial\omega}{\partial\lambda} = \frac{\partial}{\partial\lambda}\left(4\sqrt{\frac{a}{a-b}}\frac{\mathbb{K}(k)}{\sqrt{(1-\alpha)(1+\beta)}}\right), \tag{76}$$

where the last one comes from the differentiation of (63). It is very useful to simplify the above set of equations $\{(75),(76)\}$ before the actual calculation. We do this expressing the three elliptic functions $\mathbb{K}(k)\,,\mathbb{E}(k)\,,\mathbb{PI}(n,k)$ from the relations (68),(63) as

$$\begin{aligned}
\mathbb{K}(k) &= \frac{1}{4}\pi\omega\sqrt{\frac{a-b}{a}}\sqrt{(1-\alpha)(1+\beta)}\\
\mathbb{E}(k) &= \frac{\pi}{2\sqrt{a(a-b)(1-\alpha)(1+\beta)}}\left(\frac{-J_A}{\sqrt{\lambda}}+\frac{1}{2}\omega\left(b(\alpha-\beta)+a(2-\alpha+\beta)\right)\right)\\
\mathbb{PI}(n,k) &= \frac{\pi\sqrt{a(a-b)(1-\alpha)(a+\beta)}}{8ab}\left(\frac{J_B}{\sqrt{\lambda}}-b(\nu-\omega)\right),
\end{aligned} \tag{77}$$

and substituting them back to $\{(75),(76)\}$. The equations (75) becomes

$$\begin{aligned}
&b(1-\beta)\lambda\left(2J_A-\left(b(\alpha-\beta)+a(2-\alpha+\beta)\right)\sqrt{\lambda}\omega\right)\frac{\partial\alpha}{\partial\lambda}\\
&\quad-(1-\alpha)\left(b\lambda\left(2J_A+\left(b(\alpha-\beta)+a(2-\alpha+\beta)\right)\sqrt{\lambda}\omega\right)\frac{\partial\beta}{\partial\lambda}\right)\\
&\qquad-(a-b)(\alpha-\beta)(1-\beta)\left(J_B+2b\lambda^{\frac{3}{2}}\left(\frac{\partial\nu}{\partial\lambda}-\frac{\partial\omega}{\partial\lambda}\right)\right)=0
\end{aligned} \tag{78}$$

and

$$\begin{aligned}
&(1+\alpha)\lambda\left(2J_A-\left(-b(\alpha-\beta)+a(2+\alpha-\beta)\right)\sqrt{\lambda}\omega\right)\frac{\partial\beta}{\partial\lambda}\\
&\quad-(1-\beta)\lambda\left(2J_A+\left(b(\alpha-\beta)-a(2+\alpha-\beta)\right)\sqrt{\lambda}\omega\right)\frac{\partial\alpha}{\partial\lambda}\\
&\qquad-2(1+\alpha)(1-\beta)\left(J_A+\left(b(\alpha-\beta)+a(2-\alpha+\beta)\right)\lambda^{\frac{3}{2}}\right)\frac{\partial\omega}{\partial\lambda}=0
\end{aligned} \tag{79}$$

and the third one (76) turns to

$$\begin{aligned}
&\left(-1+\beta^2\right)\left(2J_A+\left(-b\alpha(\alpha-\beta)-a(2-(\alpha-\beta)\alpha)\right)\sqrt{\lambda}\omega\right)\frac{\partial\alpha}{\partial\lambda}\\
&\quad-\left(-1+\alpha^2\right)\left(2J_A-\left(-b\beta(\alpha-\beta)+a(2+(\alpha-\beta)\beta)\right)\sqrt{\lambda}\omega\right)\frac{\partial\beta}{\partial\lambda}\\
&\qquad+\left(-1+\alpha^2\right)\left(-1+\beta^2\right)2(a-b)(\alpha-\beta)\sqrt{\lambda}\frac{\partial\omega}{\partial\lambda}=0.
\end{aligned} \tag{80}$$

We find $\frac{\partial\alpha}{\partial\lambda}$ and $\frac{\partial\beta}{\partial\lambda}$ differentiating the relation (59) as

$$\begin{aligned}
\frac{\partial\alpha}{\partial\lambda} =& \frac{1}{(a-b)(\alpha-\beta)\omega^2}\left(2\kappa\frac{\partial\kappa}{\partial\lambda}+(b(\alpha-\beta)+a(\alpha+\beta))\omega\frac{\partial\nu}{\partial\lambda}\right.\\
&\left.+(b\alpha(\alpha-\beta)+a(2-(\alpha-\beta)\alpha))\omega\frac{\partial\omega}{\partial\lambda}\right)\\
\frac{\partial\beta}{\partial\lambda} =& -\frac{1}{(a-b)(\alpha-\beta)\omega^2}\left(2\kappa\frac{\partial\kappa}{\partial\lambda}+(-b(\alpha-\beta)+a(\alpha+\beta))\omega\frac{\partial\nu}{\partial\lambda}\right.\\
&\left.+(-b\beta(\alpha-\beta)+a(2+(\alpha-\beta)\beta))\omega\frac{\partial\omega}{\partial\lambda}\right). \qquad (81)
\end{aligned}$$

Substituting $\frac{\partial\alpha}{\partial\lambda}$ and $\frac{\partial\beta}{\partial\lambda}$ from (81) and the roots α and β from (60) into the set of equations $\{(78),(79),(80)\}$ we obtain a form, which includes the three terms $\frac{\partial\kappa}{\partial\lambda}$, $\frac{\partial\nu}{\partial\lambda}$, $\frac{\partial\omega}{\partial\lambda}$, the conserved charges and parameters κ,ν,ω. Solving this complicated set of equations for $\frac{\partial\kappa}{\partial\lambda}$ we find

$$W\left(-J_B\nu-J_A\omega+2\kappa\lambda^{\frac{3}{2}}\frac{\partial\kappa}{\partial\lambda}\right)=0, \qquad (82)$$

where the term W is of the following form

$$\begin{aligned}
W =& -2a^2\omega^3+\kappa^2\sqrt{(a-b)\kappa^2+ab\nu^2+a(a-b)\omega^2}\\
&+2a\omega\left(-\kappa^2+b\left(\omega^2-\nu^2\right)+\omega\sqrt{(a-b)\kappa^2+ab\nu^2+a(a-b)\omega^2}\right)\\
&+b\left(2\kappa^2\omega+(\nu-\omega)(\nu+\omega)\sqrt{(a-b)\kappa^2+ab\nu^2+a(a-b)\omega^2}\right).
\end{aligned}$$

Assuming $W\neq 0$ we find the following expression for $\frac{\partial\kappa}{\partial\lambda}$

$$\frac{\partial\kappa}{\partial\lambda}=\frac{J_B\nu+J_A\omega}{2\kappa\lambda^{\frac{3}{2}}}. \qquad (83)$$

Finally substituting $\frac{\partial\kappa}{\partial\lambda}$ from (83) to (74) we obtain

$$a_{\mathcal{L}AA}=\frac{\imath\left(\kappa^2\sqrt{\lambda}+J_B\nu+J_A\omega\right)}{4\pi^2\kappa}$$

which coincides with the solution for the three-point function (73) .

4 Conclusions

The holographic conjecture has been tested in many cases and impressive results about anomalous dimensions of the gauge theory operators, integrable structures, etc., and crucial properties of various gauge theories at strong coupling have been established. One of the main challenges ahead is to find efficient methods for calculation of the correlation functions.

While discovering a semiclassical trajectory controlling the leading contribution to the three-point correlator of arbitrary "heavy" vertex operators is so far an unsolved problem, we have seen that one can use the trajectory for the correlation function of two "heavy" operators, which is straightforward to find, to compute the correlator containing two "heavy" and one "light" states at strong coupling [17, 18]. The approach based on insertion of vertex operators was put forward in Ref. [19, 25] where the authors also suggested that the same method applies to higher n-point correlation functions with two "heavy" and n-2 "light" operators. Namely, the semiclassical expression for n-point correlator should be given by a product of "light" vertex operators calculated on the worldsheet surface determined by the "heavy" operator insertions. In the present paper we

considered string theory on $AdS_5 \times T^{1,1}$ and computed three-point correlation functions of two “heavy“ (string) and one ”light“ (super-gravity) states at strong coupling, applying the ideas of Ref. [19] for calculation of correlation functions using vertex operators for the corresponding states. We examined the method in the case of a folded string solution with one and two spins in $T^{1,1}$ part of the geometry. The solutions we use are the most simple ones and correspond to particular gauge theory operators discussed in the text. We also checked the consistency of the obtained structure constants of the correlators with those obtained as $2\pi^2 a_{\mathcal{L}AA} = \lambda\partial E/\partial\lambda$ as suggested in [19] and obtained complete agreement. This was possible because the dispersion relations for the string solutions are simple enough. Keeping parameters a and b of the $T^{1,1}$ metric free gives the option to compare the results from our considerations, namely strings in $AdS_5 \times T^{1,1}$, with the case when the geometry is $AdS_5 \times S^5$.

The correlations functions for the case of giant magnons and single spikes is much more complicated. This can be seen also just looking at the dispersion relations for these solutions [34], which give a transcendental equation for the energy. Certainly, along with the generalization of the method to include more heavy operators, the study of the correlation functions in less supersymmetric gauge theories at strong coupling remains a big challenge.

A The integrals

For completeness we first write the indefinite integrals as

$$\begin{aligned} I_1(\theta) &= \int \frac{1}{\sqrt{(\cos(\theta)-\alpha)(\cos(\theta)-\beta)}} d\theta \\ I_2(\theta) &= \int \frac{\cos(\theta)}{\sqrt{(\cos(\theta)-\alpha)(\cos(\theta)-\beta)}} d\theta \\ I_3(\theta) &= \int \sqrt{(\cos(\theta)-\alpha)(\cos(\theta)-\beta)} d\theta. \end{aligned} \tag{A.1}$$

It is then clear that the integrals I_i defined in (66) are just

$$I_i = I_i(\theta)\big|_0^{arccos(\beta)}. \tag{A.2}$$

The explicit form of the integrals (A.1) is the following

$$I_1(\theta) = \frac{-2\mathbb{F}\left(arcsin\left(\frac{(1-\alpha)(\beta-\cos\theta)}{(\beta-\alpha)(1-\cos\theta)}\right), \frac{2(\beta-\alpha)}{(1-\alpha)(1+\beta)}\right)}{\sqrt{(\alpha-1)(1+\beta)}} \tag{A.3}$$

$$\begin{aligned} I_2(\theta) =& \frac{-2}{\sqrt{(1+\beta)(1-\alpha)}}\left[\mathbb{F}\left(arcsin\left(\sqrt{\frac{1+\beta}{1-\beta}}\tan\left(\frac{\theta}{2}\right)\right), \frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right)\right. \\ &\left. +2\mathbb{PI}\left(\frac{\beta-1}{\beta+1}, -arcsin\left(\sqrt{\frac{1+\beta}{1-\beta}}\tan\left(\frac{\theta}{2}\right)\right), \frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right)\right] \\ I_3(\theta) =& \frac{1}{\sqrt{(1-\alpha)(1+\beta)}}\left[(1-\alpha)(1+\beta)\mathbb{E}\left(arcsin\left(\sqrt{\frac{1+\beta}{1-\beta}}\tan\left(\frac{\theta}{2}\right)\right), \frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right)\right. \\ &+ 2\alpha(1+\beta)\mathbb{F}\left(arcsin\left(\sqrt{\frac{1+\beta}{1-\beta}}\tan\left(\frac{\theta}{2}\right)\right), \frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right) \\ &+ 2(\alpha+\beta)\mathbb{PI}\left(\frac{\beta-1}{\beta+1}, -arcsin\left(\sqrt{\frac{1+\beta}{1-\beta}}\tan\left(\frac{\theta}{2}\right)\right), \frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right) \\ &\left. + (1-\alpha)\sqrt{(1+\beta)(\alpha-\cos\theta)(\beta-\cos\theta)}\tan\left(\frac{\theta}{2}\right)\right]. \end{aligned}$$

The integrals I_1, I_2, I_3 (A.2) thus becomes

$$
\begin{aligned}
I_1 =& \frac{2\mathbb{K}\left(\frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right)}{\sqrt{(1-\alpha)(1+\beta)}} \\
I_2 =& -2\frac{\mathbb{K}\left(\frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right) - 2\mathbb{PI}\left(\frac{\beta-1}{\beta+1}, \frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right)}{\sqrt{(1-\alpha)(1+\beta)}} \\
I_3 =& \frac{1}{\sqrt{(1-\alpha)(1+\beta)}}\left[(1-\alpha)(1+\beta)\,\mathbb{E}\left(\frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right)\right. \\
&\left. +2\alpha(1+\beta)\,\mathbb{K}\left(\frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right) + 2(\alpha+\beta)\,\mathbb{PI}\left(\frac{\beta-1}{\beta+1}, \frac{(1-\beta)(1+\alpha)}{(1+\beta)(1-\alpha)}\right)\right]
\end{aligned}
\tag{A.4}
$$

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